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Physical Operations, Diagram Operations

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Operations on diagrams are examined as part of a search for a model of superposition. Diagram operations are required to represent physical processes.

1. INTRODUCTION

A physical operation, something you do, is a physical process, something that happens. This seems obvious in a way: just that whenever you do anything, you are doing something physical. But physicists have become sloppy on this point, thinking of measurements, for example, as mathematical operations. In order to describe a measurement or any physical manipulation or operation, it is necessary not only to give a mathematical representation of the transformation, but also a description of the physical process, just how the transformation was effected physically.

If all physical processes are represented as networks of a fundamental discrete process,* then physical operations will be so represented as well. Given a network, it can be changed mathematically in many ways: links can be removed or added, connections can be switched, parts of the network can be interchanged, and so on. Physically, these operations may not be possible.

What operations are physically possible? In this paper, after a formal definition of a diagram, some operations are defined which may model physical processes. All physical operations must add interactions; none can take away interactions that already happen. The formal representation of a physical operation must result in a representation of a possible physical process.

* D. Finkelstein and others (1969–1974) in the Spacetime Code series. For references see McCollum (1978). Part of Spacetime Code VI appeared as "Discrete Feynman Diagram" in Castell et al. (1977).

The relevant mathematics for networks of discrete quantum processes is graph theory. The diagrams I will define are not strictly graphs because they are not sets of point pairs; they are sets of edges. I have considered $2n$-regular (see Section 3) diagrams because it is an easy generalization, even though I think 4-regular will give the most immediate physical results. This treatment is too rigorous for some purposes and insufficiently mathematical for others. It is intended to be a grounding rigorous enough to support future work.

I am in search of a theory that is born quantum. What does that mean? It must have both discrete and continuous characteristics. Also, it must have a superposition operation.

I am starting with a discrete object, the diagram. If there is a superposition operation, there is hope of continuity at the appropriate limit. In this paper I am concerned with the search for the superposition operation.

There are several possibilities for the way superposition will be introduced. It could have to be put in by hand such that each edge of a diagram is individually superposable. That is suspect because it is forced. Also, there arise questions about the superposition of different diagrams: surely two processes that are not elementary can be superposed. Either this would follow from the superposition of elementary processes, which is problematic mainly because there would be topological constraints; or superposition would be a regular diagram operation, arising from nonquantum diagram properties. In the latter case superposition must be definable in the usual way as a diagram operation.

Which makes it the same as the case I am exploring in this paper: If I look at diagram operations that preserve diagram properties, without making any special quantum requirements, will I find an operation that would serve as a superposition? If so, I have answered in the affirmative the question whether quantum properties arise naturally, without special requirements. If I find no such operation, the question remains open. It could be that I have overlooked something. It could be that I should go to a more complex level.

A third possibility is that superposition is not a two-diagram operation giving a third diagram, but an operation among larger sets of diagrams or something else altogether. If so, the physical interpretation of a single diagram would be significantly different than if a diagram operation superposition is found.

2. DIAGRAMS

A graph is a set of pairs of points. If an edge is a point pair, then edges with the same endpoints are the same edge. This seems like a restriction unnecessary and probably fatal to the physics. As long as I cannot use graph theory directly, I might as well define exactly what I mean. Many definitions and theorems carry over easily from graph theory anyway. The difference is this: graphs start with points, diagrams start with edges; in a diagram, a point is the end of an edge.

Definition. A *diagram* is a set of internal edges i_n along with a set of external edges e_m. An internal edge has two endpoints called vertices which may be the same. An external edge has two distinct endpoints; one is a vertex (i.e., can be the endpoint of an internal edge) and one is an external point. Each external point is the endpoint of exactly one external edge. Endpoints are said to be included in edges, two to an edge. Incidence relations among edges are described as the inclusion of common vertices.

A diagram is a representation of a physical process, perhaps still at a more fundamental level than the processes we usually observe. Most observed physical processes are incomplete, having an ignored or unknown past and future and many unmeasured aspects. So the representation of the physical process should be by nature incomplete, having room for inputs and outputs, places where other processes can continue. These are the external edges. There is a natural composition which will allow diagrams to be multiplied, representing the continuation of one process into another. Any process, even the continued existence of an object, will be represented by the connection of lines through the process. A process may change, say water to steam, but there will still be lines connecting through the process.

Definition. External edge composition is an operation between two different external edges: e_i, including vertex V_n and external point E_i; and e_j, including vertex V_m and external point E_j. The edges e_i and e_j may be of the same or of different diagrams. The result of the operation is a new internal edge with endpoints V_n and V_m. E_i and E_j are deleted.

Thus the process which was incomplete at E_i and E_j is seen to connect from V_n to V_m or vice versa.

A line may connect through a series of edges as a walk, where each edge is a step. Approximate continuity at a more complex and macroscopic level is another matter entirely. The connection of lines here may imply the conservation of various quantities, like charge and lepton number.

3. WALKS

I am not sure at this point that the physical process should be represented by a regular diagram or by a diagram with an even number of edges at each vertex, but there are strong reasons to suspect that both statements are true in the most common physical cases.

The most basic, pervasive physical properties will be found in the simplest vertex. Those are the quantum properties of discreteness and the special relativistic properties of space-time. Both quantum and relativistic properties probably depend on regularity. I say that both because of the similarity between the quantum and relativistic groups (Finkelstein, 1969–1974; Castell et al., 1977; McCollum, 1978) and because of the similarity between the motions on a vertex of four edges and those two groups (McCollum, 1978). At this point these remarks are only motivational.

Why an even number of edges at each vertex? To make it easy to model conservation (McCollum, 1978). Especially the particle conservation laws, like charge, spin, strangeness, hadron number, will be critically dependent on the possibility of defining conservation as locally as possible in the physical model. What does the number of edges have to do with that? It tells whether the diagram can be covered by walks, where a walk is just a path that connects. Without a path that connects, it would be hard to model something continuing along a path.

Both these points are germane to this section. It is awkward if not impossible to construct an operation preserving regularity with just sets of vertices and edges. The external point composition is a natural regularity-preserving operation, but the operation needs to be generalized to an operation between diagrams. On Eulerian diagrams there is a structure that leads naturally to such a generalization. The relevant definitions follow.

Definitions. (1) The *degree* of a vertex is the number of times the vertex is included in internal and external edges. (2) A diagram is *Eulerian* if, when the external edges are composed by pairs in any way, it is a Eulerian pseudograph in the sense of Harary (1969). (3) A diagram is *n-regular* if every vertex is of degree *n*.

In order to show that "Eulerian diagram" is well defined it is necessary to prove the following: If a diagram becomes a Eulerian pseudograph under one pairing of external edges, then it will become one under any pairing. This is true since the pairing and composition affect the degrees of the vertices not at all.

Since all but at most one of the external edges are composed away, whether the diagram is Eulerian depends only on the degrees of the vertices. The graph theoretical result carries over immediately: A diagram is Eulerian if and only if every one of its vertices is of even degree.

Definitions. (1) A *walk* is a sequence of edges in a diagram such that each edge ends on the same vertex the next one begins on, no edge is repeated, and the whole walk either begins and ends on the same vertex or begins and ends on external points. (2) A *covering by walks* is a set of walks over a diagram such that every edge of the diagram is included in the set once and only once.

Remarks. (1) For an n-regular Eulerian diagram, $n/2$ walks cross at each vertex, counting a walk crossing itself as two walks crossing. (2) It makes sense to speak of the external-point composition of two walks that begin and end on external points. The composition depends on order.

A number of operations between two diagrams with walks can be defined by taking compositions of walks in any well-defined way. I will define one simply using the external-point composition.

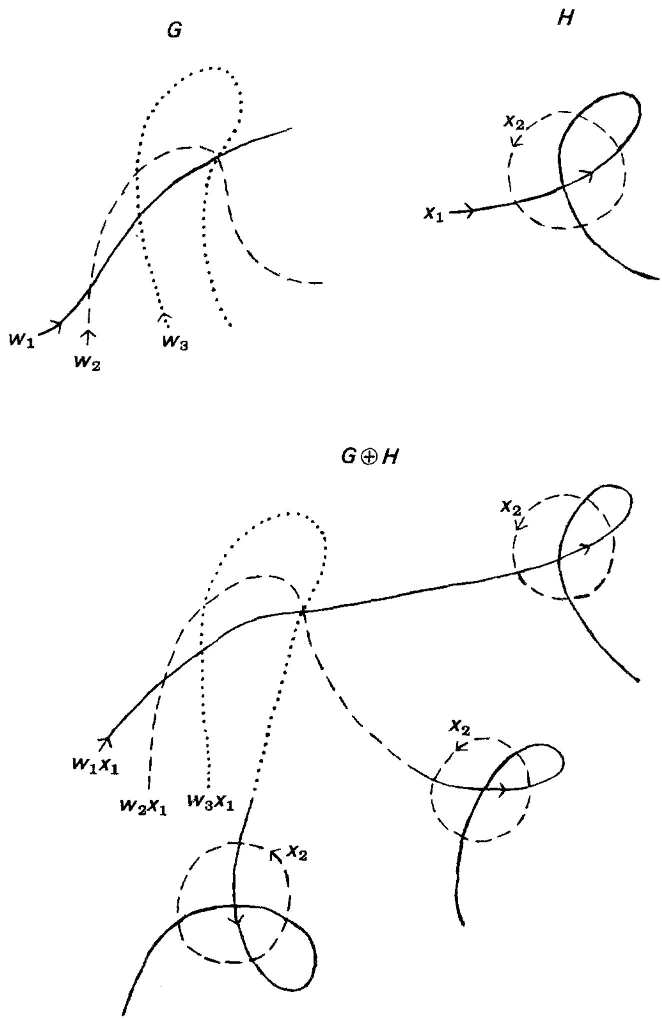

Fig. 1. G, H, and $G \oplus H$.

Definition. Let G and H be Eulerian diagrams, G be broken into walks $w_1, \ldots, w_n$, H be broken into walks $x_1, \ldots, x_m$, and $w_i x_j$ be the external point composition of walks w_i and x_j. Then $G \otimes H$ consists of all walks $w_i x_j$ composed of replicas of x_j and w_i, with all walks with j (or i) constant intersecting to form a replica of G (or H). If one or both of G and H have no walks with external points, then $G \otimes H = G \cup H$. (See Figure 1.)

We now have one regularity-preserving noncommutative operation between Eulerian diagrams with walks. To define the operation between diagrams without a specified covering by walks, all coverings must be taken. How many are there?

> *Lemma.* There are $(n-1)(n-3)\cdots 1$ ways that $n/2$ undirected walks can meet at a vertex of even degree n.

Proof. Choose an edge. Then choose another edge for it to connect to. The first choice does not count because it had to be included in some walk. The second choice out of $n - 1$ does count because it did not have to be connected to the first choice. And so on.

> *Proposition.* A Eulerian diagram with n_i vertices of degree i has $\prod_i [(i-1)(i-3)\cdots 1]^{n_i}$ coverings by undirected walks.

Proof. Given any combination of ways undirected walks can meet at each of the vertices of a Eulerian diagram, the combination determines a covering by undirected walks. So the number of coverings by undirected walks is simply the number of combinations.

To find the number of coverings by walks with direction would involve finding the number of walks in each covering. That is a harder problem.

The operation is unwieldy in the simplest case and becomes more difficult to deal with in the case where the diagrams are not covered by walks. I see no physical application at this time, but this operation is a natural one, so should be kept in mind. The following section deals with a narrowing of this operation.

4. MULTIPLICATION

In this section I want to define an operation which is more appropriate physically than the operation defined in the previous section and which leads to an interesting result.

The walk operation in the previous section can be narrowed so that only certain walks are composed. In Figure 2, only one walk is composed. The total number of vertices is constant and the number of walks has decreased by one. In that particular example, only one pair of walks from different diagrams could be composed at once while preserving the number of vertices. In Figure 1 all possible pairs are composed but the number of vertices increases.

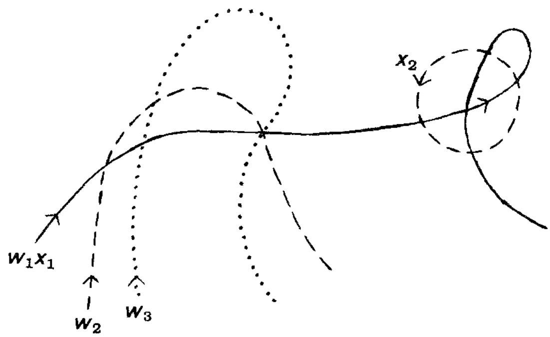

Fig. 2. Only one pair of walks of G and H from Figure 1 composed.

In this section I want to consider the operation of composing while preserving the total number of vertices. I could just specify the walks to be composed. This would presuppose the specification of walks, which is physically objectionable. Walks are a convenient tool in dealing with diagrams, but they are not observable (McCollum, 1978). The walk operation does not presuppose a choice of walks, which would be unphysical; it uses them all. Since I do not want to use them all and proliferate diagrams (Figure 1), I will define a similar operation using only the shapes of the diagrams, which are in principle observable.

The positive physical motivation for this operation is that it is the one I would use to describe one process continuing into another, like an electron continuing to be an electron or one object metamorphosing into another. The definition without walks leads to a result which may be important later in dealing with repeating processes, with the question, what makes an object an object?

This operation uses all the information that is obtainable through experiment and no more.

There are also formal reasons to define this operation. I think this operation will be the diagram analog of the multiplication of amplitudes: when the diagrams are multiplied, so are the amplitudes. This is in direct analogy to standard quantum mechanics and is discussed in Section 6 of McCollum (1978). I expect it to be analogous to multiplication of operators in a Lie group.

So I will call this operation diagram multiplication. It is the most natural diagram operation. But it is not particularly easy to specify. In fact, the whole point of defining the operation is in the exact specification of how one process continues into another. There are many ways, all interesting, but a disaster if done at random. For example, a repetition might not be a repetition at all if the multiplication is done differently each time. It may be later I will want a partially specified multiplication, which will allow a certain amount of randomness.

As it is, multiplication is not defined on all pairs of diagrams: it is partially defined. Preliminary definitions and notation are necessary.

Definitions. The diagram G_s is a *subdiagram* of the diagram G if there exists a diagram G_q such that a certain pairwise composition of external edges between G_s and G_q is isomorphic to G. G_q is the *quotient* of G_s divided into G; it may not be unique unless external points or vertices are specified. (See Figure 3.)

Notation. $G_s\{E_n\}$ is a subdiagram of G means G_s is a subdiagram of G such that $\{E_n\}$ of G_s are not to be composed in forming G with any G_q. (See Figure 3.)

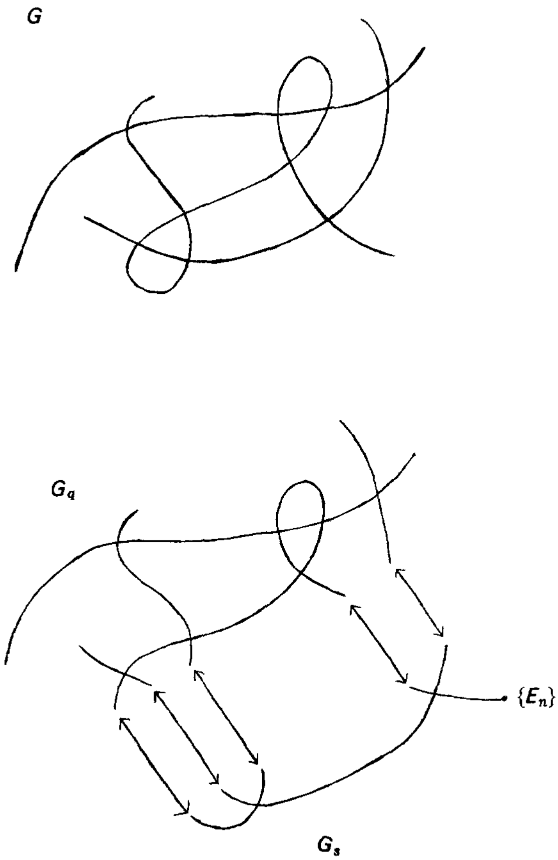

Fig. 3. Factoring a diagram. New G.

Definition. Let $G\{E_n\}$ be a subdiagram of G_s and $G_s\{E_n\}$ be a subdiagram of $T_{G,G_s\{E_n\}}G_p$. Then define $T_{G,G_s\{E_n\}}$ to be an operator multiplying G with another diagram, say G_p, such that $T_{G,G_s\{E_n\}}G_p$ is a diagram which can be obtained from G and G_p by external edge composition with the above subdiagrams. (See Figure 4.)

Question. On which diagrams is $T_{G,G_s\{E_n\}}$ defined? In a physical model this would amount to the question: Which processes G_p can be continued into the process G, if this operation is used to describe the continuation of one process into another?

In order to answer the question I need another definition.

Definition. If $G_s\{E_n\}$ is a subdiagram of G, an *image* of $G_s\{E_n\}$ in G is a subset of G such that internal edges of G_s map to internal edges of the image, external edges $\{e_n\}$ with external points $\{E_n\}$ map to external edges in the image, all other external edges map to edges, and incidences at vertices of $G_s\{E_n\}$ are preserved. (In other words, G_s and the image are isomorphic if the incidence relations of external points other than $\{E_n\}$ are discounted.) (See Figure 5.)

Answer. Let G_q be the quotient of $G_s\{E_n\}$ divided into $T_{G,G_s\{E_n\}}G_p$. Then G_q is a subdiagram of G_p. $T_{G,G_s\{E_n\}}G_p$ is defined only for those G_p's in which the images of G_q are equivalent to each other under automorphism of G_p. In other words, the product is defined if it really does not matter which side of G_p you stick G onto. (See Figures 4 and 6.)

Now that the question of when the multiplication is defined has been answered, the way is open to approach the question: When can the multiplication operation repeat indefinitely? When is $T^n_{G,G_s\{E_n\}}$ defined? Physically

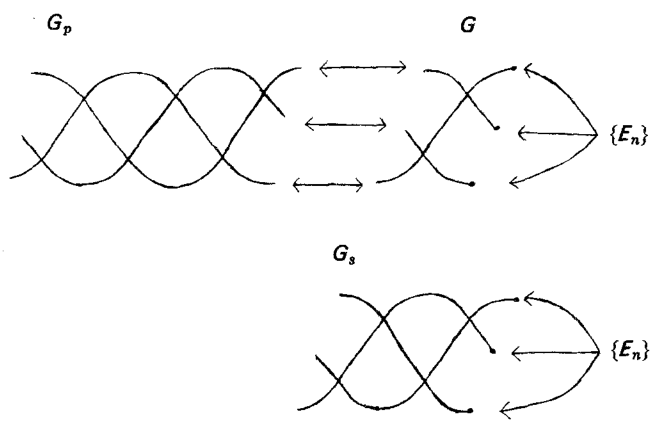

Fig. 4. Components of $T_{G,G_s\{E_n\}}G_p$.

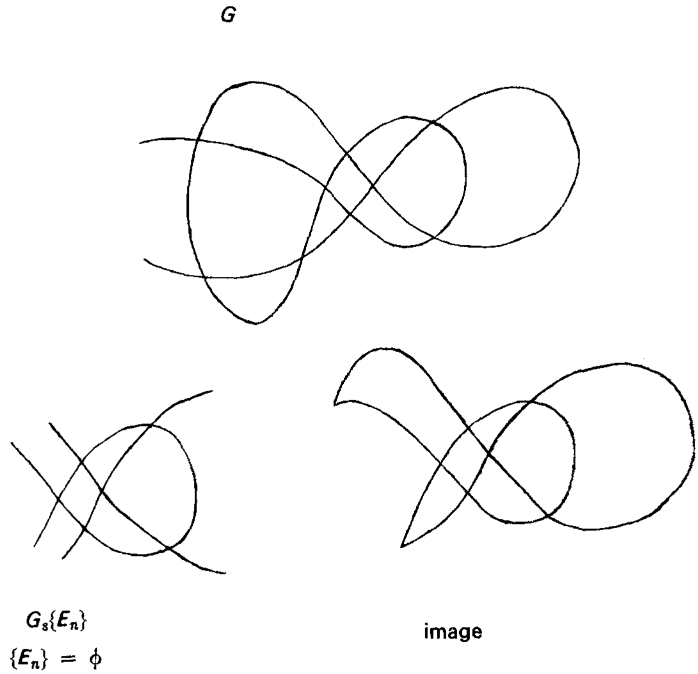

Fig. 5. $G_s\{E_n\}$ would be isomorphic to the image if incidence relations among external points of $G_s\{E_n\}$ were not considered.

the question amounts to this: If this operation is used to describe the continuation of one process into another, when can a process be described as repeating indefinitely? This question will be intimately related to such questions as: Which processes are objects? How can an object be produced? Why is time one dimensional?

The last question anticipates the result: G can be attached indefinitely in the same way only if there are at most two ends of G_p at which G can be attached.

Preliminaries. Let $\{E_m\}$ be the external points of G_p composed to make $T_{G,G_s\{E_n\}}G_p$ and G_a be the quotient of G_q divided into G_p. Then $G_a\{E_m\}$ is a subdiagram of G_p. (See Figure 6.)

> *Proposition.* Consider finite diagrams G_p. $T^k_{G,G_s\{E_n\}}G_p$ is defined for any ^{k} if and only if there are one or two images of $G_a\{E_m\}$ in $T_{G,G_s\{E_n\}}G_p$, one is disjoint from G_p, and in the case of two images the images are equivalent under automorphism of $T_{G,G_s\{E_n\}}G_p$ and there are two equivalent images of $G_a\{E_m\}$ in $T^2_{G,G_s\{E_n\}}G_p$.

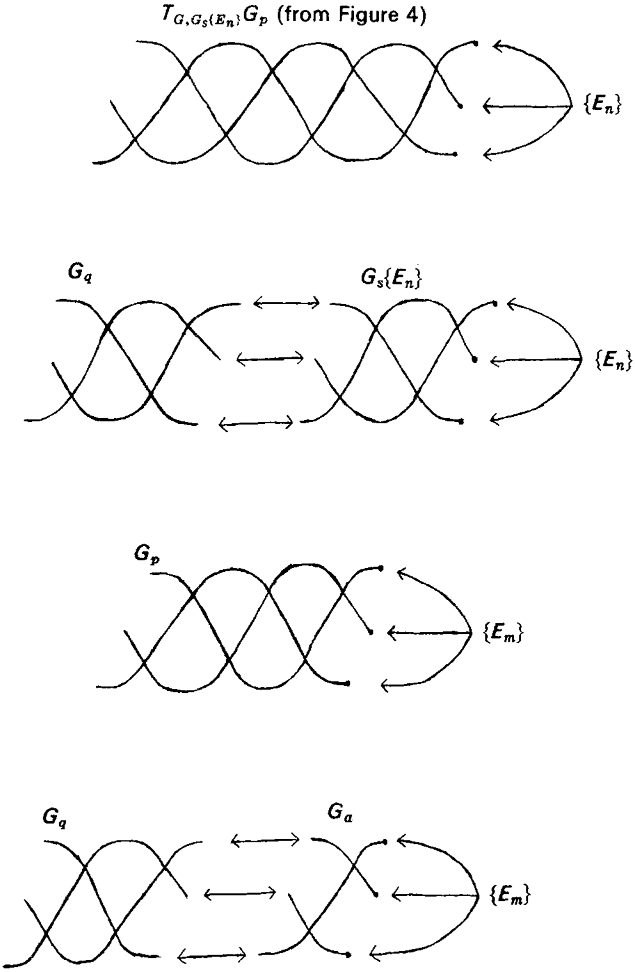

Fig. 6. G_q is a subdiagram of G_p.

Proof. If: The image of $G_a\{E_m\}$ is just the place on G_p to which G will be attached. If one image of $G_a\{E_m\}$ in $T_{G,G_s\{E_n\}}G_p$ is disjoint from G_p, then it is in G. So every time $T_{G,G_s\{E_n\}}$ iterates, it attaches an image of $G_a\{E_m\}$ while composing one away. If the images are equivalent in the first and second powers in the case of two images, the symmetry is such that they will be for all odd and even powers. In the case of one image, there is no equivalence to worry about for any power.

Only if: Suppose there are more than two images. Then either they are not equivalent in $T_{G,G_s\{E_n\}}G_p$ or they would not be after one more iteration.

Suppose both images or part of both are contained in G_p. Then G_p is a central core that is being used up, so k cannot be arbitrarily large.

This theorem specifies exactly when the multiplication operation can repeat indefinitely. If a physical theory is based on diagram operations, this will severely limit what kinds of objects there are, that is, what kinds of processes repeat. Also it limits time to one dimension if time is defined by the direction(s) of persistence of objects.

What other operations are available? Multiplication and operations of the walk type are the best I have come up with. In the following section I discuss graph operations, the obvious candidates for generalization to diagrams.

5. GRAPH THEORETIC OPERATIONS

Diagrams are similar to graphs. The main difference is that the definition of a diagram is as a set of edges rather than as a set of pairs of points. There are other differences, like the naturalness of external edges, external-edge composition, and multiplication, but they are in principle minor: those differences would be easy to incorporate into graph theory.

The similarity leads to the hope that concepts from graph theory will be useful in diagrams, and that theorems and proofs can be converted without difficulty.

In the case of the common graph theoretic operations (Harary, 1969), the usefulness of the diagram analogs seems limited. They can be redefined to map two diagrams into a diagram, that is, to have only one external edge containing each external point. But of the four operations, only the union preserves regularity. A discussion of graph operations is, nevertheless, appropriate in this context; they are possible operations. I will define each operation and for the last three give examples of nonpreservation of regularity.

Preliminaries. Let diagrams G_1 and G_2 have disjoint internal edge sets $\{i_n\}_1$ and $\{i_n\}_2$, disjoint external edge sets $\{e_n\}_1$ and $\{e_n\}_2$, disjoint vertex sets $\{V_n\}_1$ and $\{V_n\}_2$, and disjoint external point sets $\{E_n\}_1$ and $\{E_n\}_2$. Incidence relations are given by the vertices.

Union. $G_1 \cup G_2$ has internal edge set $\{i_n\}_1 \cup \{i_n\}_2$, external edge set $\{e_n\}_1 \cup \{e_n\}_2$, vertex set $\{V_n\}_1 \cup \{V_n\}_2$, and external point set $\{E_n\}_1 \cup \{E_n\}_2$.

The union preserves regularity. It just is not very interesting. But it might be useful.

Join. $G_1 + G_2$ is the same as $G_1 \cup G_2$ but with internal edge set $\{i_n\}_1 \cup \{i_n\}_2 \cup$ {all lines connecting $\{V_n\}_1$ with $\{V_n\}_2$}. (See Figure 7.)

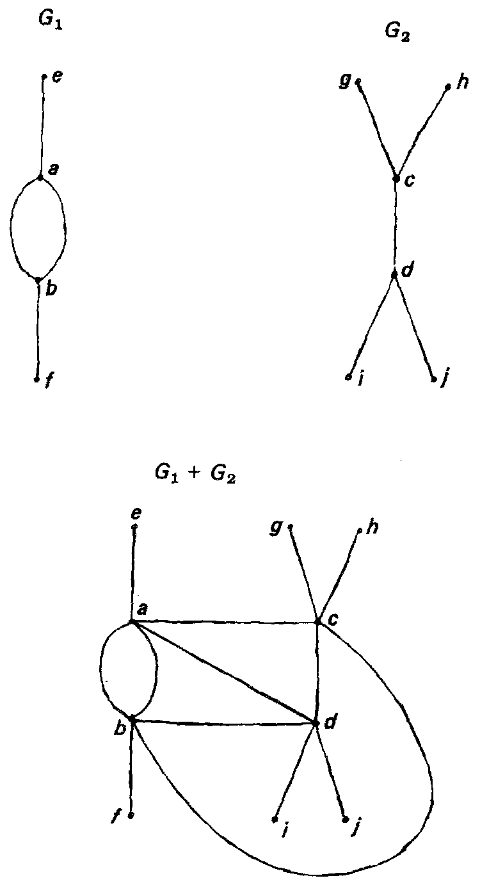

Fig. 7. Join. The external points are e, f, g, h, i, j.

Graph product. $G_1 \times G_2$ has vertex set the Cartesian product $\{V_n\}_1 \times \{V_n\}_2$ and external point set $\{E_n\}_1 \times \{V_n\}_2 \cup \{V_n\}_1 \times \{E_n\}_2$. If X and Y are either vertices or external points: when $X_1 = Y_1$ there is one edge connecting (X_1, X_2) and (Y_1, Y_2) for every edge connecting X_2 and Y_2; when $X_2 = Y_2$ there is one edge connecting (X_1, X_2) and (Y_1, Y_2) for every edge connecting X_1 and Y_1. (See Figure 8.)

Composition. $G_1[G_2]$ has the same sets of vertices and external edges as the graph product. The edge sets are the same with the addition to the internal edge set of an edge connecting (X_1, X_2) and (Y_1, Y_2) for every edge connecting X_1 and Y_1 when $X_2 \neq Y_2$ and X_1, X_2, Y_1, and Y_2 are all vertices. (See Figure 9.)

Definitions analogous to those of the graph theoretic product and

$G_1 \times G_2$

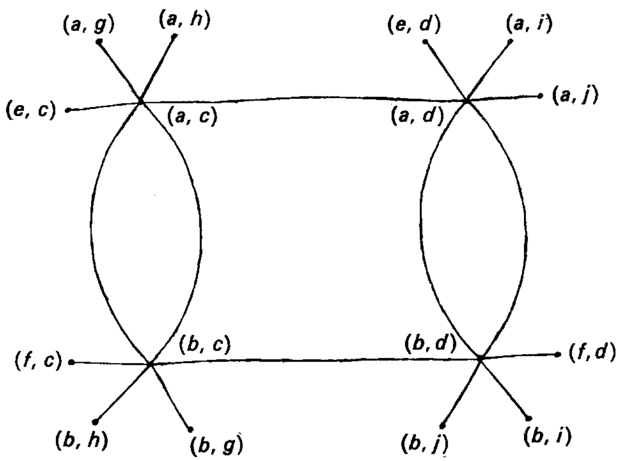

Fig. 8. Graph product. G_1 and G_2 from Figure 7.

$G_1[G_2]$

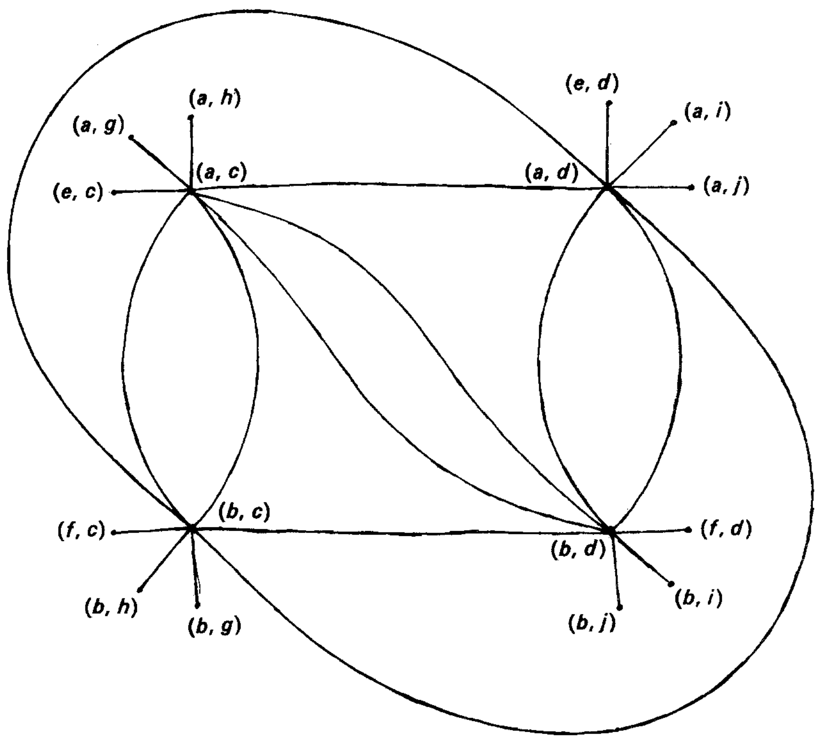

Fig. 9. Composition. G_1 and G_2 from Figure 7.

composition can be made in several different ways. They all fail to preserve regularity in much the same way these do.

These do not seem to be interesting physical operations. There is, in particular, no superposition here.

6. WHERE IS SUPERPOSITION?

The main result of this paper is negative: no operation like superposition has been found, even though it would be expected to be between simple processes, such as the operations which have been discussed.

What is superposition? A superposition operation commutes, preserves regularity, takes two physical processes and gives a physical process, is associative, and has nice properties in relation to the amplitude, yet to be defined. The requirements that it be commutative and preserve regularity eliminate all the operations that I have mentioned but one: the union. The problem with the union is that it lacks physical motivation and leads nowhere.

So I have failed to find a superposition operation among the operations I have considered. It seems unlikely that one is possible that preserves regularity; such an operation would probably involve external-edge composition like the walk operations or multiplication, so that it would not be commutative.

What does this mean for the physical model? It means either superposition must be assumed for the tetrad, with all its properties, or I must look at a higher level for a superposition operation. The former is an unacceptable alternative from the point of view of making a coherent physical model; if I start at the bottom putting things in by hand, the resulting structure will be pretty meaningless. Then I must look at a higher level.

What does this mean for the physical model? It means that some of the quantum properties are not as basic as I thought. The duality between discrete and continuous is probably fundamental. But superposition and amplitude are not. The uncertainty principle probably is not, although indeterminacy certainly is, in the sense that what cannot be measured is indeterminate. Then quantum mechanics as we know it will be derived from the fundamental indeterminacy and our habits of measurement.

7. RESULTS

As far as superposition is concerned, the result is an imperative to look for a higher-level operation.

Two physically interesting operations have been defined. The walk operation is one in a family of operations of the same type. All such operations tend to yield much bigger diagrams than the two diagrams the operation was performed on. How much bigger depends on the number of walks in the original diagrams. In Section 3 the number of possible walks is found.

The other operation is multiplication. The main result of the paper is the theorem about multiplication: it only iterates in one dimension. This result will probably be central to the definition of objects and time.

REFERENCES

Castell, L., Drieschner, M., and von Weizsaecker, C. F., Eds. (1977). *Quantum Theory and the Structures of Time and Space* Vol. II, Carl Hanser Verlag, Munich.

Finkelstein, David. (1969–1974). "The Spacetime Code series.

Harary, Frank. (1969). *Graph Theory*, Addison-Wesley Pub. Co., Reading, Massachusetts.

McCollum, Gin. (1978). "Spacetime Code: Preliminaries and Motivations," in *Found. Phys.*, **8**, 211.